Generalized negligible morphisms and their tensor ideals

Or: How zero is zero?

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Dimensions in monoidal categories

- Let $\mathcal{C}$ be a monoidal rigid spherical category whose *Hom* spaces are k-vector spaces over a field k, $End(\mathbf{1}) \cong k$.
- For any object can define the trace function Tr_X on $End(X)$.
- The dimension of X is then $Tr(id_X) \in End(\mathbf{1}) \cong k$.
- If $\mathcal{C}$ is semisimple, then $\dim(X) \neq 0$ for all indecomposable X.
- If $\mathcal{C}$ is not semisimple, it has a largest proper tensor ideal

$$\mathcal{N}(X, Y) = \{f \in Hom(X, Y) \mid Tr(f \circ g) = 0 \; \forall g : Y \to X\},$$

  the ideal of negligible morphisms.
- The associated thick ideal is

$$N = \{X \mid X \text{ indecomposable }, \dim(X) = 0.\}$$

- Observation: Often times this categorial dimension is zero. Aim: Introduce a measure for for the "nullity" of the dimension.

Examples

- $Tilt(U_q(\mathfrak{g}))$ ($\mathfrak{g}$ a semisimple Lie algebra), the category of tilting modules for (Lusztig's) quantum group at a primitive ℓ-th root of unity q over $Q(q)$ or $\mathbb{C}$.
- $Tilt(G)$, the category of tilting modules for semisimple and simply connected G over $\mathbb{F}_p$ (or its algebraic closure).
- In both cases indecomposable tilting modules are parametrized by X^+, the dominant integral weights. Their categorial dimension vanishes iff λ is not in the fundamental alcove (for ℓ and p bigger than h).
- $Rep(GL_n)$, Deligne's interpolating category for the parameter $n \in \mathbf{Z}$. Indecomposable objects are parametrized by bipartitions (λ^L, λ^R). The categorial dimension vanishes iff $length(\lambda^L) + length(\lambda^R) > |n|$.

Let $\mathcal{C}$ be a monoidal category. A *tensor ideal* $\mathcal{I}$ in $\mathcal{C}$ consists of a subgroup $\mathcal{I}(X, Y) \subset Hom(X, Y)$ for all $X, Y \in \mathcal{C}$ such that

- for all $X, Y, Z, W \in \mathcal{C}$ and $f \in Hom(X, Y)$ and $h \in Hom(Z, W)$

$$f \in \mathcal{I}(Y, Z) \text{ implies } f \circ g \in \mathcal{I}(X, Y) \text{ and } h \circ f \in \mathcal{I}(Y, W);$$

- $f \in Hom(X, Y)$ implies $id_Z \otimes f \in \mathcal{I}(Z \otimes X, Z \otimes Y)$ and likewise from the right.

A collection of objects I in a monoidal category $\mathcal{C}$ is called a *thick ideal* of $\mathcal{C}$ if the following conditions are satisfied:

(i) $X \otimes Y \in I$ whenever $X \in \mathcal{C}$ and $Y \in I$.

(ii) If $X \in \mathcal{C}$, $Y \in I$ and there exist $\alpha : X \to Y$, $\beta : Y \to X$ such that $\beta \circ \alpha = id_X$, then $X \in I$.

To any tensor ideal $\mathcal{I}$ we can associate the thick ideal I given by

$$I = \{X \in \mathcal{C} \mid id_X \in \mathcal{I}(X, X)\}.$$

I-negligible morphisms

- Let R be a local ring with maximal ideal $\mathfrak{m}$. We assume $\mathcal{C}(R)$ to be a monoidal rigid spherical tensor category whose *Hom* spaces are free R-modules.
- We call a morphism $f : X \to Y$ I-negligible if $Tr_X(g \circ f) \in I$ and $Tr_Y(f \circ g) \in I$ for all morphisms $g : Y \to X$. An object X is called I-negligible if $Tr_X(a) \in I$ for all $a \in End(X)$.
- Then the I-negligible morphisms form a tensor ideal $\mathcal{N}_I$ in $\mathcal{C}(R)$ and the I-negligible objects form a thick ideal N_I.
- Special case: $I = \mathfrak{m}^k$. Then we use the notation $\mathcal{N}_k$ and N_k.

Mod $\mathfrak{m}$ evaluations

- If M is a free R-module of rank r, we obtain a well-defined vector space $M/\mathfrak{m}M$ over $k = R/\mathfrak{m}$ of dimension r. We call the mod $\mathfrak{m}$ evaluation $\mathcal{C}$ of $\mathcal{C}(R)$ the category $\mathcal{C}$ whose objects are in 1-1 correspondence with the ones of $\mathcal{C}(R)$, and where $Hom(X, Y) = Hom_R(X, Y)/\mathfrak{m}Hom_R(X, Y)$.
- The images of $\mathcal{N}_I$ and N_I define tensor ideals respectively thick ideals in $\mathcal{C}$.
- Example: N_1 are the indecomposable negligible objects, i.e. $\dim_{\mathcal{C}}(X) = 0$.
- Get a chain of thick ideals $N_1 \supset N_2 \supset N_3 \subset \ldots$ and tensor ideals $\mathcal{N}_1 \supset \mathcal{N}_2 \supset \mathcal{N}_3 \subset \ldots$ in any mod $\mathfrak{m}$ evaluation as well as quotient functors $\mathcal{C}/\mathcal{N}_3 \to \mathcal{C}/\mathcal{N}_2 \to \mathcal{C}/\mathcal{N}$.
- The N_k are "hidden" in $\mathcal{C}$ and only become visible when viewing $\mathcal{C}$ as a mod $\mathfrak{m}$ evaluation.
- Question: What can we say about the N_k and why are they interesting?
- For $X \in \mathcal{C}$ we say X has nullity k if $X \in N_k$ and k is minimal with this property.

Mod I evaluations - examples

Theorem

1. *Various Deligne categories over $\mathbb{C}$ are mod $\mathfrak{m}$ evaluations from their analogs over the completion of $\mathbb{C}[t]_{(t-n)}$, the polynomial ring localized at $(t-n)$, i.e. all rational functions over $\mathbb{C}$ which are evaluable at $t=n$.*
2. *$Tilt(U_q(\mathfrak{g}))$ is the mod $\mathfrak{m}$ evaluation of $Tilt(U_q(\mathfrak{g}))_R$ where R is (the completion of) $\mathbb{C}[v]_{(v-q)}$.*
3. *$Tilt(G)$ is the mod $\mathfrak{m}$ evaluation of $Tilt(G)_{\mathbf{Z}_p}$ over the p-adic integers $\mathbf{Z}_p$.*

Proof: Canonical/Crystal bases and Kempf vanishing over $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$; Lifting of primitive idempotents in towers of algebras.

Example: The categorial dimension in $Tilt(G)$ is the p-dimension, the usual dimension reduced mod p. The maximal ideal in $\mathbf{Z}_p$ is $p\mathbf{Z}_p$, so $dim_{Tilt(G)}(X) = 0$ iff p divides $dim\ X$ (vector space dimension). So are we measuring the p-divisibility of $dim\ X$?

The modular SL_2-case

- For SL_2 the tilting modules are parametrized by $\mathbb{N}$, $T(0)$, $T(1)$, $T(2)$,...
- Introduce $St_r = L((p^r - 1)\rho)$ and let $I_r = < St_r >$.
- For $SL(2)$ the I_r are a complete list of thick ideals. A tilting module $T(m)$ is in I_r if and only if $m \geq p^r - 1$.
- For $p > 2$

$$T(\lambda) \in N_k \text{ if and only if } p^k | \dim T(\lambda)$$

  where dim refers to the dimension of $T(\lambda)$ as a vector space. In other words, N_k measures the p-divisibility of the dimension of $T(\lambda)$.
- It is important to assume $p > 2$ here. Indeed the dimensions of the first tilting modules in the $p = 2$ case are

$$\dim T(0) = 1, \ \dim T(1) = St_1 = 2, \dim T(2) = 4, \ \dim T(3) = St_3 = 4.$$

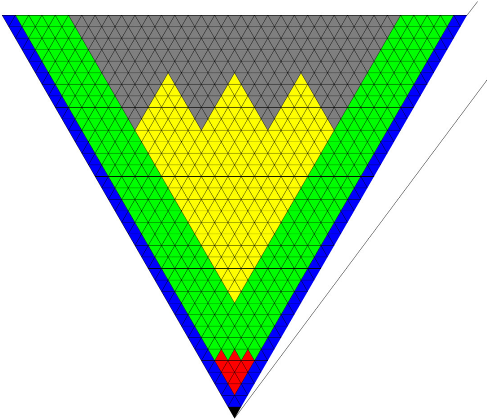
- Although $\dim T(2) = 4$, it is not in N_2. Over $\mathbf{Z}_2$ we have $Tr(id_{T(2)}) = 4$, but we can write $T(2) \cong T(1) \otimes T(1)$. Hence there is an endomorphism f of $T(2)$ which permutes the two factors. It is easy to see that $Tr(f) = 2$, hence the trace is not always contained in $(2)^2$ and so $T(2) \notin N_2$.

The tilting cases A_1 and A_2

- For Sl_3 every ideal is k-negligible for some k (picture is taken from Andersen *Cells in affine Weyl groups* for $p \geq 5$).
- A complete list of thick ideals is:
- The Steinberg ideals with nullity $3s$ for $s = 1, 2, \ldots$ and
- for $s = 0, 1, 2, \ldots$ the ideals generated by the $T(\lambda)$ with

$$(\lambda + \rho, \rho) = p^{s+1}, \ (\lambda + \rho, \alpha_1) = rp^s, 1 \leq r < p$$

of nullity $3s + 1$.



The example *Rep* GL_t

- Let λ be a Young diagram. (i,j) denotes the box in the i-th row and j-th column of λ.
- Let $h(i,j)$ be the length of the hook whose northwest corner is the box (i,j).
- Let $R(\lambda^L, \lambda^R)$ denote the indcomposable object corresponding to the bipartition (λ^L, λ^R). Then

$$\dim(R(\lambda, 0)) = P_\lambda(t) = \prod_{(i,j)\in\lambda} \frac{t-i+j}{h(i,j)} \text{ for } \mathit{Rep}\ GL_t \text{ over } R.$$

- Zeros of $P_\lambda(T)$ are precisely the integers $-j+i$ for $(i,j) \in \lambda$. For the partition $\lambda = (k^{n+k})$ the polynomial has exactly k zeros for $t = n$, i.e.

$$P_{k^{n+k}}(t) = (t-n)^k \cdot \mathit{something}.$$

- Hence for each $k \in \mathbb{N}$ there must be an N_k. In fact:

Theorem

Every thick ideal resp. every tensor ideal is of the form N_k *resp.* $\mathcal{N}_k$*. These form a strictly decreasing chain of ideals. The same is true for the Rep* O_n*-case.*

- For the thick ideals this gives a new classification of the thick ideals in *Rep* GL_t and *Rep* O_t.
- For the tensor ideals this is based on results by Coulembier.
- Should be true for the quantum versions as long as q is not a root of unity.

Recall that for any objects $X, Y \in \mathcal{C}$ and any endomorphism $f \in X \otimes Y$ we have the left trace $t_L(f) \in End_{\mathcal{C}}(X)$ and the right trace $t_R(f) \in End_{\mathcal{C}}(Y)$.

Definition

If I is a thick ideal in $\mathcal{C}$ then a *trace on* I is a family of linear functions

$$\{\mathrm{t}_V : End_{\mathcal{C}}(V) \to R\}$$

where V runs over all objects of I and such that following two conditions hold.

1. If $U \in I$ and $W \in \mathcal{C}$ then for any $f \in End_{\mathcal{C}}(U \otimes W)$ we have
$$\mathrm{t}_{U \otimes W}(f) = \mathrm{t}_U(tr_R(f)).$$
2. If $U, V \in I$ then for any morphisms $f : V \to U$ and $g : U \to V$ in $\mathcal{C}$ we have
$$\mathrm{t}_V(g \circ f) = \mathrm{t}_U(f \circ g).$$

- Assume that the maximal ideal $(p) \subset R$ is generated by the element p.
- Let I be a tensor ideal all of whose objects are k-negligible, e.g. the ideal N_k of all k-negligible objects. Then we define the modified trace $Tr_X^{(k)}$ and modified dimension $\dim^{(k)}(X)$ for an object X in I by ($a \in End(X)$)

$$Tr_X^{(k)}(a) \;=\; \frac{1}{p^k}\; Tr_X(a), \qquad \dim^{(k)}(X) \;=\; \frac{1}{p^k}\; \dim(X),$$

Note that this is well-defined since $Tr_X(a) \in (p)^k \; \forall a \in End(X)$. It is clear that $Tr_X^{(k)}(id_X) \; = \dim^{(k)}(X)$.

Lemma

Let X, Y be objects in I, and let Z be an object in $\mathcal{C}$. Then we have
(a) $Tr_X^{(k)}(ab) = Tr_Y^{(k)}(ba)$ for all morphisms $a: X \to Y$ and $b: Y \to X$,
(b) $Tr_{X\otimes Y}^{(k)}(a \otimes c) = Tr_X^{(k)}(a) Tr_Z(c)$ and $\dim^{(k)}(X \otimes Y) = \dim^{(k)}(X) \dim(Y)$ for $a \in End(X)$, $c \in End(Z)$.

Gives a *modified* trace on I in the sense of Geer, Kujawa, Patureau-Mirand,...
In particular: All ideals in Deligne categories and for $Tilt(\ldots)$ have nontrivial modified traces.

- Any link L with m components can be obtained as the closure of a braid β. For chosen objects $X_1, \ldots, X_m$ obtain

$$\Phi(\beta) \in End(X_1^{\otimes c_1} \otimes X_2^{\otimes c_2} \otimes \ \ldots \otimes X_m^{\otimes c_m}).$$

- The link invariant $\mathcal{L}^{(X_1, \ \ldots, X_m)}(L)$ is then defined by

$$\mathcal{L}^{(X_1, \ \ldots, X_m)}(L) = Tr(\Phi(\beta)).$$

Theorem

(a) If the object $\mathbf{X}^{\otimes \mathbf{m}}$ is k-negligible, then we obtain a new link invariant $\mathcal{L}^{(X_1, \ \ldots, X_m),(k)}$ defined by

$$\mathcal{L}^{(X_1, \ \ldots, X_m),(k)}(L) = \frac{1}{p^k} \mathcal{L}^{(X_1, \ \ldots, X_m)}(L)$$

which is well-defined and yields an invariant with values in $R/(p)$.
(b) If $R = \widehat{\mathbb{C}[v]}_{(v-q)}$ and $p = v - q$, then $R/(p) \cong \mathbb{C}$ and the value of the $R/(p)$-valued invariant is equal to $k! \frac{d^k}{dq^k} \mathcal{L}^{(X_1, \ \ldots, X_m)}(L)_{|v=q}$, which is valid for its evaluation on any m-component link L.

How do the tensor ideals look like in the quantum case?

- Have a system of hyperplanes on $\mathfrak{h}^*$ from the orbits of the generating hyperplanes under the affine Weyl group. They can be described explicitly by

$$H_{\alpha,k} \;=\; \{x \in \mathfrak{h}^*,\ (x,\alpha) = k\ell\}, \quad \alpha \in \Delta_+, k \in \mathbf{Z},$$

  if $d|\ell$.
- These hyperplanes make $\mathfrak{h}^*$ into a cell complex as follows: We call an intersection of k hyperplanes maximal if it has dimension $n-k$, and we denote by $\mathfrak{h}^*(n-k)$ the union of all maximal intersections of k hyperplanes.
- The set of j-cells then is given by all connected components of $\mathfrak{h}^*(j)\backslash\mathfrak{h}^*(j-1)$, with $\mathfrak{h}^*(-1)$ being the empty set.
- Call the n-cells *alcoves*, and lower-dimensional cells *facets*. The $(n-1)$-cells which are in the closure of a given alcove A are called the *walls* of A.

Quantum ideals

The following theorem gives an explicit description of all thick ideals in quantum $U_q(\mathfrak{sl})_n)$. In this case Ostrik constructed thick ideals corresponding to two-sided cells in the affine Weyl group. These cells are parametrized by partitions λ of n. To each λ we associate a facet $F_0(\lambda)$.

Theorem

The thick ideal $\mathcal{I}(\lambda) = \mathcal{I}(F_0(\lambda))$ generated by the tilting modules $T(\nu)$ for which $\nu + \rho \in F_0(\lambda)$ coincides with the thick ideal constructed by Ostrik for the cell in the dominant Weyl chamber corresponding to the two-sided cell labeled by the partition λ^T. The nullity of any generating module $T(\nu)$ of that ideal is equal to the value of Lusztig's a-function of that cell.

Remark: The thick ideal N_k is the sum of the $I(\lambda)$ (λ partition of n) for which the nullity is $\geq k$.
We have an analogous conjecture for modular type A.

Open questions

- Extension to more general categories? E.g. small quantum group?
- Classify thick ideals for Deligne categories at roots of unity.
- Currently we define modified traces only if the maximal ideal has one generator. It would be interesting to define modified traces if the maximal ideal is not principal.
- Understand the relation between the nullity and the *a*-function in the quantum case.