Pseudorandomness properties of the Liouville function

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The Liouville function $\lambda : \mathbb{N} \to \{-1, +1\}$ is the unique completely multiplicative function that equals $-1$ at every prime.

In other words, $\lambda(1) = 1$ and $\lambda(pn) = -\lambda(n)$ for all primes $p$ and natural numbers $n$.

Equivalently, $\lambda(n) = 1^{\Omega(n)}$, where $\Omega(n)$ is the number of prime factors of $n$ (counting multiplicity).

The Liouville function is a close cousin of the Möbius function $\mu$, and most of the discussion in this talk about $\lambda$ has a counterpart for $\mu$. But we focus on $\lambda$ for simplicity.
First few values of $\lambda$:

\[ 1, -1, -1, 1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1,
The study of this function is guided by the following informal heuristic principle:

**Liouville pseudorandomness principle**

The Liouville function should behave statistically like a random function from \( \mathbb{N} \) to \( \{-1, +1\} \).

There is also a closely related *Möbius pseudorandomness principle* in the literature.
For instance, this principle predicts decay in the ordinary averages

$$\mathbb{E}_{n \leq X} \lambda(n) := \frac{\sum_{n \leq X} \lambda(n)}{\sum_{n \leq X} 1}$$

and the logarithmic averages

$$\mathbb{E}_{n \leq X}^{\log} \lambda(n) := \frac{\sum_{n \leq X} \frac{\lambda(n)}{n}}{\sum_{n \leq X} \frac{1}{n}}.$$ 

- Elementary arguments give $$\mathbb{E}_{n \leq X}^{\log} \lambda(n) \ll \frac{1}{\log X}.$$  
- **Prime number theorem:** $$\mathbb{E}_{n \leq X} \lambda(n) = o(1).$$  
- **Riemann hypothesis:** $$\mathbb{E}_{n \leq X} \lambda(n) \ll_{\varepsilon} X^{-1/2+\varepsilon}$$ for any $$\varepsilon > 0.$$
The prime number theorem is usually proven by complex-analytic methods, exploiting Dirichlet series identities such as
\[ \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} \]
together with properties of the Riemann zeta function \( \zeta(s) \).

However, in (McNamara 2020, Richter 2020) a new proof was given that principally relied on the multiplicativity \( \lambda(pn) = -\lambda(n) \) at small primes \( p \).

The arguments give significantly weaker bounds than standard existing proofs, but are elementary and extend to more general contexts.

We illustrate the method by first proving the very weak estimate
\[ \mathbb{E}_{n \leq X} \log n \lambda(n) = o(1). \]
A useful tool in this regard is

**Elliott’s inequality (Elliott, 1979)**

Let \( f : I \rightarrow \mathbb{C} \) be a function on an interval \( I \) of length at least 1. Then

\[
\sum_{p \leq |I|^{0.1}} \frac{1}{p} |\mathbb{E}_{n \in I} f(n) - \mathbb{E}_{n \in I : p | n} f(n)|^2 \ll \mathbb{E}_{n \in I} |f(n)|^2.
\]

This is proven from the Cauchy–Schwarz inequality (or large sieve inequality) and the fact that the functions \( p 1_{p | n} - 1 \) are almost orthogonal on \( I \).
Informally, Elliott’s inequality asserts that one has the sampling law
\[ \mathbb{E}_{n \in I : p | n} f(n) \approx \mathbb{E}_{n \in I} f(n) \]
for “most” small primes \( p \leq |I|^{0.1} \).

The error term in this approximation is not very strong, because
\[ \sum_{p \leq |I|^{0.1}} \frac{1}{p} \sim \log \log |I| \] is not very large.
In terms of ordinary averages, Elliott’s inequality asserts that
\[ \mathbb{E}_{n \leq X} f(n) \approx \mathbb{E}_{n \leq X/p} f(pn) \]
for most small primes \( p \ll X^{0.1} \).

For logarithmic averages, the distinction between \( X \) and \( X/p \) gets erased:
\[ \mathbb{E}_{n \leq X}^{\log} f(n) \approx \mathbb{E}_{n \leq X}^{\log} f(pn) \]
for most small primes \( p \ll X^{o(1)} \).

This “approximate dilation invariance” makes logarithmic averages particularly tractable.
Now we can give a quick proof of the weak inequality

\[ \mathbb{E}_{n \leq X} \log \lambda(n) = o(1). \]

From Elliott’s inequality we have

\[ \mathbb{E}_{n \leq X} \log \lambda(n) \approx \mathbb{E}_{n \leq X} \log \lambda(pn) \]

for most small primes \( p \leq X^{o(1)}. \)

On the other hand, from the multiplicativity property \( \lambda(pn) = -\lambda(n) \) we have

\[ \mathbb{E}_{n \leq X} \log \lambda(pn) = -\mathbb{E}_{n \leq X} \log \lambda(n). \]

Combining these two facts give the proof.
The same argument, when applied to ordinary averaging and iterated, gives

\[ E_{n \leq X/p_1 \cdots p_k} \lambda(n) \approx (-1)^k E_{n \leq X} \lambda(n) \]

for most small primes \( p_1, \ldots, p_k \) and bounded \( k \).

One can now prove the prime number theorem through one of two related strategies:

(Richter) Find nearby numbers \( p_1 \cdots p_k \approx p'_1 \cdots p'_k \) with \( k, k' \) having distinct parity.

(McNamara) Find either a prime \( p \) or semiprime \( p_1 p_2 \) close to a location \( P \) where \( E_{n \leq X/P} \lambda(n) \) experiences a sign change.
One can use these types of methods to gain qualitative control of correlations of the Liouville function \( \lambda \) with other functions. We give a simple example here. In 1937, Davenport proved

**Theorem (Davenport, 1937)**

For any real \( \alpha \), we have

\[
\mathbb{E}_{n \leq X} \lambda(n) e(-\alpha n) \ll_A \log^{-A} x
\]

for any \( A > 0 \). (Here \( e(\theta) := e^{2\pi i \theta} \).)

Here we prove the weaker estimate

\[
\mathbb{E}_{n \leq X} \lambda(n) e(-\alpha n) = o(1).
\]
Suppose for contradiction that

\[ |\mathbb{E}_{n \leq X} \lambda(n) e(-\alpha n)| \gg 1. \]

By Elliott's inequality, this implies that

\[ |\mathbb{E}_{n \leq X} \lambda(n) e(-\alpha n)| \gg 1 \]

for most small scales \( P \) and most primes \( p \sim P \).

By multiplicativity \( \lambda(pn) = -\lambda(n) \), this implies that

\[ |\mathbb{E}_{n \leq X} \lambda(n) e(-\alpha pn)| \gg 1. \]
The relation \( |E_{n \leq X/P} \lambda(n)e(-\alpha pn)| \gg 1 \) is a correlation property which we write informally as

\[
\lambda(n) \approx_{n \sim X/P} e(\alpha pn).
\]

Pretending correlation is transitive, we then conclude that

\[
e(\alpha pn) \approx_{n \sim X/P} e(\alpha p'n)
\]

for many primes \( p, p' \sim P \). (This can be formalised using Cauchy–Schwarz.)

Exponential sum estimates then force

\[
\alpha p = \alpha p' + O\left(\frac{P}{X}\right) \mod 1
\]

for many primes \( p, p' \sim P \).
A standard lemma of Vinogradov reveals that in order for the approximate equation

$$\alpha p = \alpha p' + O\left(\frac{P}{X}\right) \mod 1$$

to hold for many \( p, p' \sim P \), the frequency \( \alpha \) must be major arc in the sense that

$$\alpha = \frac{a}{q} + O\left(\frac{1}{X}\right) \mod 1$$

for some integers \( a, q = O(1) \).

In this major arc case we have \( e(\alpha n) \approx e(an/q) \), and we are reduced to showing that

$$\mathbb{E}_{n \leq X} \lambda(n)e(-an/q) = o(1).$$

But this follows from the **prime number theorem in arithmetic progressions** (which can also be proven by similar methods).
Similar techniques were introduced by Montgomery–Vaughan (1977), Daboussi–Delange (1982), Katai (1986), and Bourgain–Sarnak–Ziegler (2013) to control correlations of bounded multiplicative functions with various other functions of “minor arc” type.

For instance, they can be used to prove the estimate

$$\mathbb{E}_{n \leq X} \lambda(n)e(-P(n)) = o(1)$$

uniformly for all polynomials $P$ of fixed degree, or more generally

$$\mathbb{E}_{n \leq X} \lambda(n)F(g(n)\Gamma) = o(1)$$

for any nilsequence $F(g(n)\Gamma)$ (which we will not define here).

The latter estimate (with error term improved to $O_A(\log^{-A} X)$) can be used to obtain asymptotics for linear equations in primes (Green–T. 2010, 2012, Green–T.–Ziegler 2012).
Given a function $f : \mathbb{N} \rightarrow \{-1, 1\}$, we let $s_f(k)$ denote the number of possible length $k$ sign patterns $(f(n+1), \ldots, f(n+k)) \in \{-1, 1\}^k$ exhibited infinitely often by $f$, thus $1 \leq s_f(k) \leq 2^k$. We say that $f$ is deterministic if $s_f(k) = e^{o(k)}$. We have the following conjectures, in increasing order of strength:

**Weak Sarnak conjecture (Sarnak 2012)**

$\lambda$ is not deterministic. In other words, $s_\lambda(k) \gg e^{ck}$ for some $c > 0$.

**Logarithmically averaged Sarnak conjecture**

For any deterministic $f : \mathbb{N} \rightarrow \{-1, 1\}$, we have

$$\mathbb{E}_{n \leq X} \lambda(n)f(n) = o(1).$$

**Strong Sarnak conjecture (Sarnak 2012)**

For any deterministic $f : \mathbb{N} \rightarrow \{-1, 1\}$, we have

$$\mathbb{E}_{n \leq X} \lambda(n)f(n) = o(1).$$
Conjecturally, $s_\lambda(k) = 2^k$, i.e., all sign patterns occur infinitely often. Known progress so far:

- $s_\lambda(k) = 2^k$ for $k \leq 4$, and $24 \leq s_\lambda(5) \leq 32$ (Hildebrand 1986, Matomäki–Radziwiłł–T. 2016, T.–Teräväinen, 2019).
- $s_\lambda(k) \geq 2k + 14$ for $k \geq 5$ (T.–Teravainen, 2019).
- $s_\lambda(k) \gg k^2$ (McNamara, 2019, building upon Frantzikinakis–Host 2018).
- $\tilde{s}_\lambda(k) \gg_A k^A$, where $\tilde{s}_\lambda(k)$ counts the length $k$ patterns that occur at least once (Matomäki–Radziwiłł–Teräväinen–T.–Ziegler, 2020).
Progress on these problems has been tied to the Chowla conjecture:

**Logarithmically averaged Chowla conjecture**

For distinct $h_1, \ldots, h_k$, one has
\[
\mathbb{E}_{n \leq X} \lambda(n + h_1) \ldots \lambda(n + h_k) = o(1).
\]

**Chowla conjecture (Chowla 1965)**

For distinct $h_1, \ldots, h_k$, one has
\[
\mathbb{E}_{n \leq X} \lambda(n + h_1) \ldots \lambda(n + h_k) = o(1).
\]

Informally, these conjectures assert that the Liouville function is statistically random on short intervals; they imply $s_\lambda(k) = 2^k$. It is known that the Chowla conjecture implies the Sarnak conjecture (Sarnak 2012), and that the logarithmically averaged Chowla conjecture is equivalent to the logarithmically averaged Sarnak conjecture (T. 2016).
Known progress on these conjectures:

- For $k = 1$, the logarithmically averaged Chowla conjecture is elementary, and the Chowla conjecture is equivalent to the prime number theorem.

- The logarithmically averaged Chowla conjecture is true for $k = 2$ (T. 2016; quantitative refinement by Helfgott–Radziwiłł 2021). A generalisation of this claim can be used to resolve the **Erdős discrepancy problem** (T. 2016).

- The logarithmically averaged Chowla conjecture is true for odd $k$ (T.–Teräväinen 2018).

- The above results also hold for the Chowla conjecture at *almost all* scales $X$ (T.–Teräväinen 2019; this application was first raised in Gomilko–Kwietnak–Lemańczyk 2018).

- Resolved in function fields when the field has high prime power order (Sawin–Shusterman 2018). Further function field analogues known (Klurman–Mangerel–Teräväinen 2020).
Recall that Elliott's inequality, proven via Cauchy–Schwarz, implies an approximation

$$E_{n \leq X} \log f(n) \approx E_{n \leq X} \log f(pn)$$

for most primes $p \leq X^{o(1)}$.

As it turns out, there is a useful variant approximation

$$E_{n \leq X} \log f(n+ph_1) \ldots f(n+ph_k) \approx E_{n \leq X} \log f(pn+ph_1) \ldots f(pn+ph_k)$$

for most primes $p \leq \log^{o(1)} X$ (T., 2016, Teräväinen–T. 2018).

This is proven by Shannon entropy inequalities rather than Cauchy–Schwarz.
Very rough sketch of proof:

- Suppose that

\[ \mathbb{E}_{n \leq X} \log f(n + ph_1) \ldots f(n + ph_k) \not\approx \mathbb{E}_{n \leq X} \log f(pn + ph_1) \ldots f(pn + ph_k), \]

then \( f(n + ph_1) \ldots f(n + ph_k) \) is not independent of the value of \( n \) mod \( p \).

- This implies, for suitable \( H \), that there is some mutual information between the random variables \( (f(n + 1), \ldots, f(n + H)) \) and \( n \) mod \( p \), when \( n \) is drawn randomly using the distribution from \( \mathbb{E}_{n \in X} \log \).

- But by the Chinese remainder theorem, the random variables \( n \) mod \( p \) are essentially independent for \( p \leq \log o^{(1)} X \).

- Entropy inequalities then give the desired contradiction.

The restriction to \( p \leq \log o^{(1)} X \) is very annoying! Recently relaxed for \( k = 2 \) by Helfgott–Radziwiłł (2021) by expander graph techniques.
Specialising this approximation

\[ \mathbb{E}_{n \leq X} \log f(n + ph_1) \ldots f(n + ph_k) \approx \mathbb{E}_{n \leq X} \log f(pn + ph_1) \ldots f(pn + ph_k) \]

to the Liouville function, we get

\[ \mathbb{E}_{n \leq X} \log \lambda(n + ph_1) \ldots \lambda(n + ph_k) \approx (-1)^k \mathbb{E}_{n \leq X} \log \lambda(n + h_1) \ldots f(n + h_k) \]

and thus

\[ \mathbb{E}_{n \leq X} \log \lambda(n + h_1) \ldots f(n + h_k) \approx (-1)^k \mathbb{E}_{p \sim P} \mathbb{E}_{n \leq X} \log \lambda(n + ph_1) \ldots \lambda(n + ph_k) \]

for many scales \( P \ll \log o(1) X \). This allows us to control Chowla-type correlations by more tractable averages.
For instance, this approximation gives

\[ \mathbb{E}_{n \leq X}^\log \lambda(n)\lambda(n+1) \approx \mathbb{E}_{p \sim P} \mathbb{E}_{n \leq X}^\log \lambda(n)\lambda(n+p). \]

The right-hand side can be controlled using the circle method by simpler expressions such as

\[ \mathbb{E}_{n \leq X}^\log |\mathbb{E}_{h \leq P} \lambda(n+h)|^2. \]

This in turn can be controlled by a powerful result of Matomäki and Radziwiłł (2016) from multiplicative number theory.

This technique settles the \( k = 2 \) case of the log-averaged Chowla’s conjecture.
For odd $k$, the approximation gives

$$
E_{n \leq X} \lambda(n+h_1) \ldots \lambda(n+h_k) \approx -E_{p \sim P} E_{n \leq X} \lambda(n+ph_1) \ldots \lambda(n+ph_k)
$$

and

$$
E_{n \leq X} \lambda(n + h_1) \ldots \lambda(n + h_k)
$$

$$
\approx E_{p_1 \sim P} E_{p_2 \sim P} E_{n \leq X} \lambda(n + p_1 p_2 h_1) \ldots \lambda(n + p_1 p_2 h_k)
$$

for various scales $P, P_1, P_2$.

On the other hand, higher order versions of the circle method give

$$
E_{p \sim P} E_{n \leq X} \lambda(n + ph_1) \ldots \lambda(n + ph_k)
$$

$$
\approx E_{p_1 \sim P} E_{p_2 \sim P} E_{n \leq X} \lambda(n + p_1 p_2 h_1) \ldots \lambda(n + p_1 p_2 h_k).
$$

This gives the log-averaged Chowla’s conjecture for odd $k$. 

For general $k$, these arguments (together with higher order Fourier analysis) reduce the log-averaged Chowla conjecture to

**Higher order uniformity conjecture**

If $H = H(X)$ goes to infinity as $X \to \infty$, then

$$\mathbb{E}_{n \leq X} \sup_g |\mathbb{E}_{h \leq H} \lambda(n + h)F(g(n)\Gamma)| = o(1)$$

for any nilmanifold $G/\Gamma$ and continuous function $F : G/\Gamma \to \mathbb{C}$, where $g : \mathbb{Z} \to G$ ranges over polynomial sequences.

The simplest unsolved case of this conjecture is

**Fourier uniformity conjecture**

If $H = H(X)$ goes to infinity as $X \to \infty$, then

$$\mathbb{E}_{n \leq X} \sup_{\alpha} |\mathbb{E}_{h \leq H} \lambda(n + h)e(\alpha h)| = o(1).$$
Current state of progress:

- The weaker “sup outside” statement
  \[ \sup_{\alpha} \mathbb{E}_{n \leq X} |\mathbb{E}_{h \leq H} \lambda(n + h) e(\alpha h)| = o(1) \] is known for any \( H \) going to infinity (Matomäki–Radziwiłł-T., 2015). Similarly for higher order (He–Wang, 2019).

- The Fourier uniformity is known for \( H \geq X^\epsilon \) (in fact \( H \geq \exp(\log^{1-\delta} X) \)) (Matomäki–Radziwiłł-T., 2020; simplified proof, Walsh 2021). Similarly for higher order (Matomäki–Radziwiłł–Teräväinen–T.–Ziegler 2020).

- The stronger assertion
  \[ \sup_{\alpha} |\mathbb{E}_{h \leq H} \lambda(n + h) e(\alpha h)| \ll_A \log^{-A} x \] is known for \( H \geq X^{2/3+\epsilon} \) (Matomäki–Shao, 2019). The methods should also extend to higher order.


Reaching \( H = \log^{o(1)} X \) would resolve the (log-averaged) Chowla conjecture!
Heuristic sketch of the local $H \geq X^\varepsilon$ Fourier uniformity result (in the simplified arrangement of Walsh):

- Suppose for contradiction that

$$\mathbb{E}_{n \leq X} \sup_{\alpha} |\mathbb{E}_{h \leq H} \lambda(n + h)e(-\alpha h)| \gg 1.$$ 

- Informally, this means that for many $x \sim X$, one has a correlation

$$\lambda(n) \approx e(\alpha_x n)$$

for $n \in [x, x + H]$, and some frequency function $x \mapsto \alpha_x$. 
\[ \lambda(n) \approx e(\alpha x n) \text{ for } n \in [x, x + H] \]

- Using the multiplicativity \( \lambda(pn) = -\lambda(n) \approx \lambda(n) \), we conclude that
  \[ \lambda(n) \approx e(\alpha_x pn) \]
  for \( n \in [x/p, x/p + H/p] \) and small primes \( p \asymp P \) for some scale \( P \ll H \). Since we also expect
- Since we also expect \( \lambda(n) \approx e(\alpha_{px} n) \) in this range, we are led to the following approximate scaling law:
  \[ p\alpha_x = \alpha_{x/p} + O(P/H) \text{ mod 1} \]
  when \( x \asymp X \) and \( p \ll P \).
Similar arguments predict a local stability

$$\alpha_x = \alpha_{x'} + O(1/H) \mod 1$$

when $x \sim X$ and $x' = x + O(H)$.

These sorts of relations can be formalised (in a statistical sense) by various Cauchy–Schwarz based methods, such as Elliott’s inequality and the large sieve inequality.

We are now faced with a question of arithmetic combinatorical flavour: can we describe the functions $x \mapsto \alpha_x$ from $[1, X]$ to $\mathbb{R}/\mathbb{Z}$ which obey the scaling law

$$p\alpha_x = \alpha_{x/p} + O(P/H) \mod 1$$

for $x \sim X$ and $p \sim P$, as well as the above local stability law?
\[ p\alpha_x = \alpha_x/p + O(P/H) \mod 1 \text{ for } x \asymp X, p \asymp P \]
\[ \alpha_x = \alpha_{x'} + O(1/H) \mod 1 \text{ for } x \asymp X, x' = x + O(H) \]

- An “obvious” solution to the above system of equations is given by taking \( \alpha_x \) to be of the form

\[ \alpha_x = \frac{T}{x} + O\left(\frac{x}{HX}\right) \mod 1 \]

for all \( x \leq X \) and some \( T = O(X^2/H^2) \).

- In this case, we have from Taylor expansion that

\[ e(\alpha_x n) \approx n^{iT} \]

and hence

\[ \lambda(n) \approx n^{iT} \]

on most short intervals \([x, x + H]\). But this can be ruled out by the theorems of Matömaki and Radziwiłł.

- So it remains to show that the obvious solutions are the only solutions.
There are two key difficulties in solving the system of equations

\[ p\alpha_x = \alpha_x/p + O(P/H) \text{ mod 1 for } x \asymp X, p \asymp P \]

\[ \alpha_x = \alpha_{x'} + O(1/H) \text{ mod 1 for } x \asymp X, x' = x + O(H) \]

(i) One is working modulo 1, which makes operations such as dividing by \( p \) ambiguous.

(ii) The implied graph connecting \( x + O(H) \) to \( x/p + O(H/P) \) for \( x \asymp X, p \asymp P \) is quite sparse as there are not many primes \( p \) involved. So it is difficult to directly relate \( \alpha_x, \alpha_y \) for two different \( x, y \asymp X \).
The equations

\[ p\alpha_x = \frac{\alpha_x}{p} + O(P/H) \mod 1 \text{ for } x \asymp X, \ p \sim P \]
\[ \alpha_x = \alpha_{x'} + O(1/H) \mod 1 \text{ for } x \asymp X, \ x' = x + O(H) \]
describe functions \( x \mapsto \alpha_x \) at scales \( x \asymp X \) and \( x \asymp X/P \).

Walsh’s idea is to use this data to construct an additional function \( x \mapsto \alpha_x \) at scale \( x \asymp XP \) such that

\[ p\alpha_x = \frac{\alpha_x}{p} + O(1/H) \mod 1 \text{ for } x \asymp XP, \ p \sim P. \]

Iterating, he constructs functions \( x \mapsto \alpha_x \) at scales \( x \asymp XP^k \) with

\[ p_1 \ldots p_k \alpha_x = \frac{\alpha_x}{p_1 \ldots p_k} + O(1/H) \mod 1 \text{ for } x \asymp XP^k, \ p \sim P. \]

For \( P \geq X^\varepsilon \), we can make \( P^k \) larger than \( X \), and the graph associated to this equation is now “dense” enough to conclude the desired structure

\[ \alpha_x = \frac{T}{X} + O\left(\frac{1}{H}\right) \mod 1 \text{ for } x \asymp X. \]
How does Walsh get from

\[ p^\alpha x = \alpha x/p + O(P/H) \mod 1 \text{ for } x \asymp X, p \asymp P \]

\[ \alpha x = \alpha x' + O(1/H) \mod 1 \text{ for } x \asymp X, x' = x + O(H) \]

to

\[ p^\alpha x = \alpha x/p + O(1/H) \mod 1 \text{ for } x \asymp XP, p \sim P. \]

- For any \( x \asymp XP \), the existing equations give

\[ q^\alpha x/p = p^\alpha x/q + O(P/H) \mod 1 \text{ for } p, q \asymp P. \]

- For any single pair of distinct \( p, q \), one can use the Chinese remainder theorem to solve this equation to find \( \alpha x \in \mathbb{R}/\mathbb{Z} \) such that

\[ \alpha x/p = p^\alpha x + O(1/H) \mod 1; \quad \alpha x/q = q^\alpha x + O(1/H) \mod 1. \]

- Separation properties of the Farey fractions \( a/p, b/q \) can be used to ensure that \( \alpha x \) is essentially independent of \( p, q \).
In conclusion:

- We now have a reasonably good understanding of the Liouville function on short intervals \([x, x + H], x \asymp X\) so long as \(H \gg X^\varepsilon\) (where the graph connecting \(x + O(H)\) to \(x/p + O(H/p)\) for \(p \ll H\) essentially has bounded diameter).

- If we could get down to \(H = \log^{o(1)} X\) we would be able to resolve the Chowla and Sarnak conjectures (in their logarithmically averaged form).

- However a new technique seems needed to bridge the “log barrier” \(H = \log X\). The Matomäki–Radziwiłł theorem crosses this barrier, but it relies heavily on multiplicativity and the method has so far has not been extended to the local uniformity conjecture.
Thanks for listening!